Noncommutativity and the Aharonov-Bohm Effect

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The possibility of detecting noncommutive space relics is analyzed by using the Aharonov-Bohm effect. If space is noncommutative, it turns out that the holonomy receives kinematical corrections that tend to diffuse the fringe pattern. This fringe pattern has a non-trivial energy dependence and, therefore, one could observe noncommutative effects by modifying the energy of the incident electrons beam in the Tonomura experimental arrangement.

I. INTRODUCTION

There are arguments in string theory suggesting that spacetime could be non-commutative [1]. Although this property might be an argument in favor of new renormalizables effective field theories [2], it represents also a trouble because we need to explain the transition between the commutative and noncommutative regimes.

If the noncommutative effects are important at very high energies, then one could posit a decoupling theorem that produces the standard quantum field theory as an effective field theory and that does not remind the noncommutative effects. However, our experience in atomic and molecular physics [3] strongly suggests that the decoupling is never complete, and the high energy effects appear in the effective action as topological remnants [4].

Following this idea we would like to consider an example, related to topological aspects, where the appearance of noncommutative effects could be relevant. A natural candidate is the Aharonov-Bohm effect [5], where we know that the line spectrum does not depend on the relativistic nature of the electrons [6]. However, as will see below, the spatial noncommutativity smears the spectral lines and this effect, in principle, could be observed by increasing the energy of the electrons in the Aharonov-Bohm experiment.

The article is organized as follows; in section 2, we discuss the noncommutative Aharonov-Bohm effect following a very simple perturbative approach and a generalized formula for the holonomy is given. In section 3, we discuss the physical consequences of our result and we argue how to observe noncommutative effects from the Tonomura experiment. Finally, an appendix discussing the fringe pattern in the relativistic Aharonov-Bohm effect is also considered.

II. THE NONCOMMUTATIVE AHARONOV-BOHM EFFECT

Let us start considering a spinless electron moving on a two dimensional noncommutative plane $\Re^2 - \{0\}$, where $\{0\}$ is constructed putting a solenoid with a magnetic field concentrated along the x_3 axis. Since in the noncommutative case space is a collection of cells, the reader will notice that this rearrangement is just an approximation valid for small values of the anticommutative θ parameter.

The field tensor in the noncommutative plane is

$$\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ieA_\mu \star A_\nu - ieA_\nu \star A_\mu, \qquad (1)$$

where $\star$ is the Moyal product defined as

$$\mathbf{A} \star \mathbf{B}(\mathbf{x}) = e^{\frac{i}{2}\theta^{ij}\partial_i^{(1)}\partial_j^{(2)}} \mathbf{A}(\mathbf{x}_1)\mathbf{B}(\mathbf{x}_2)|_{\mathbf{x}_1=\mathbf{x}_2=\mathbf{x}}, \qquad (2)$$

and we will use $\theta^{ij} = \epsilon^{ij}\theta$.

As we are assuming that the noncommutative effects are small, one can expand the Moyal product retaining only the linear term in θ, *i.e.*

$$\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e\theta\epsilon^{\alpha\beta}\partial_\alpha A_\mu \partial_\beta A_\nu. \qquad (3)$$

Then, we must construct a gauge potential such that the magnetic field B vanishes everywhere except at the origin. In order to do that, we proceed as in the commutative case, *i.e.* from

$$\mathbf{A} = \frac{-x_2\hat{x}_1 + x_1\hat{x}_2}{x_1^2 + x_2^2}.\frac{\phi}{2\pi} \qquad (4)$$

where ϕ is the flux inside the solenoid. We construct the non-commutative potential by means of the Ansatz

$$A_1 = -x_2 f(r^2).\frac{\phi}{2\pi},$$
$$A_2 = x_1 f(r^2).\frac{\phi}{2\pi}. \qquad (5)$$

Since we have taken an expansion of the Moyal product in eq. (3), the ordinary product in (5) must be understood. Then, as in the commutative case, we impose $B_3 = \hat{F}_{12} = 0$ outside the solenoid implying the condition

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$$2f + 2r^2 f^{'} - e\theta(f^2 + 2r^2 f f^{'}) = 0, \tag{6}$$

where $f^{'} = df/dr^2$.

This differential equation can be integrated easily and yields the following solution

$$\begin{aligned} f &= \frac{1}{e\theta} \pm \frac{1}{e\theta}\sqrt{1 - \frac{e\theta}{r^2}} \\ &= \frac{1}{e\theta} \pm \frac{1}{e\theta}[1 - \frac{e\theta}{2r^2} - \frac{e^2\theta^2}{8r^4} + ...]. \end{aligned} \tag{7}$$

From (7) we see that the commutative limit is smooth for the minus sign in the above equation *i.e.*

$$f = \frac{1}{2r^2} + \frac{e\theta}{8r^4} + ... \tag{8}$$

and the potential

$$\begin{aligned} A_1 &= -x_2\left(\frac{1}{2(x_1^2 + x_2^2)} + \frac{e\theta}{8(x_1^2 + x_2^2)^2} + ...\right).\frac{\phi}{2\pi} \\ A_2 &= x_1\left(\frac{1}{2(x_1^2 + x_2^2)} + \frac{e\theta}{8(x_1^2 + x_2^2)^2} + ...\right).\frac{\phi}{2\pi}, \end{aligned} \tag{9}$$

describes a magnetic field zero everywhere except at the origin. This potential is also the non- commutative generalization of the magnetic monopole for θ small [7].

Next step is to solve the Schrödinger equation for a particle with mass m moving in the field (9). However, instead of doing this we remind that, in the commutative case, the Schrödinger equation in an external gauge potential is solved by

$$\psi = e^{ie\int_C dx^j A_j}\varphi, \tag{10}$$

where the $U(1)$ holonomy $e^{ie\int_C dx^j A_j}$, in general, is a non-integrable factor, *i.e.* dependent on the path C, being φ the free solution of the Schrödinger equation.

However, although formally (10) solve the Schrödinger equation, the holonomy involves in a non trivial way the dynamics of the gauge potential and it hides all the complications of $\mathbf{A}$. Our goal below will be to find an approximate expression for the holonomy for θ small.

Let us suppose that the operator $D_j = -i\partial_j + eA_j$ satisfies the eigenvalue equation

$$D_j \star \psi = k_j \psi. \tag{11}$$

Then, using (11) the Schrödinger equation becomes

$$\hat{H}\psi = \frac{1}{2m} D_j * D_j * \psi = \frac{1}{2m} k_j k_j \psi. \tag{12}$$

In order to solve (11) we use the Ansatz

$$\psi = e^F, \tag{13}$$

and therefore, for θ small

$$\begin{aligned} D_j\psi &= -i\partial_j e^F + eA_j * e^F \\ &= e^F[-i\partial_j F + eA_j + \frac{i}{2}e\theta\epsilon^{lm}(\partial_l A_j)(\partial_m F)] \end{aligned}$$

and furthermore

$$-i\partial_j F + eA_j + \frac{i}{2}e\theta\epsilon^{lm}(\partial_l A_j)(\partial_m F) = k_j. \tag{14}$$

Now, one can solve (14) perturbatively by expanding F and A_j in powers of θ, *i.e.*

$$F = F^{(0)} + \theta F^{(1)} + ... \tag{15}$$

$$A_j = A_j^{(0)} + \theta A_j^{(1)} + ..., \tag{16}$$

then at zero order in θ, equation (14) gives

$$-i\partial_j F^{(0)} + eA_j^{(0)} = k_j, \tag{17}$$

and the following expression for $F^{(0)}$ is obtained

$$F^{(0)} = ik_j(x - x_0)_j - ie\int_{x_0}^{x} dx_j A_j^{(0)}. \tag{18}$$

The first term in the RHS is just the free particle solution if we interpret k_j as the wave number and the second term is the $U(1)$ holonomy for the commutative case. Thus, at zero order, we reproduce the commutative solution of the Schrödinger equation.

If we retain first order terms in θ, the following differential equation is obtained

$$-i\partial_j F^{(1)} + eA_j^{(1)} + \frac{i}{2}e\theta\epsilon^{lm}(\partial_l A_j^{(0)})(\partial_m F^{(0)}) = 0, \tag{19}$$

and the integration of (19) gives

$$F^{(1)} = -e\int_{x_0}^{x} dx_j A_j^{(1)} - \frac{ie\theta}{2}\int_{x_0}^{x} dx_j \epsilon^{ml}(k_m - eA_m^{(0)})\partial_l A_j^{(0)}). \tag{20}$$

The first term in the RHS is an additive correction to the commutative holonomy, *i.e.*

$$-e\int_{x_0}^{x} dx_j (A_j^{(0)} + \theta A_j^{(1)}) =: -e\int_{x_0}^{x} dx_j A_j, \tag{21}$$

and a second one include a velocity dependent term which can be written as follows[1]:

$$-\frac{i}{2}e\int_{x_0}^{x} dx_j\ \epsilon^{ml}k_m\partial_l A_j^{(0)} = -\frac{i}{2}em\int dx_j\ (\mathbf{v} \times \nabla A_j^{(0)})_3, \tag{22}$$

[1]A different contribution was obtained in [8]. However, the mistake of these authors was corrected after our paper appeared in a the subsequent version of [8]. We would like to thank to the referee for drawing our attention on this point.

For the third term our calculation yields to

$$\int_{x_0}^{x} dx_j \, (\mathbf{A}^{(0)} \times \nabla A_j^{(0)})_3. \tag{23}$$

Thus, at this order in θ, the nonconmutative holonomy is

$$\begin{aligned}\mathcal{W}(x,\, x_0\,) = \\ &= \exp\Bigg[-ie\int_{x_0}^{x} dx_j A_j - \frac{i}{2}me\theta\int_{x_0}^{x} dx_j\,[(\mathbf{v}\times\nabla A_j^{(0)})_3 \\ &- e(\mathbf{A^{(0)}}\times\nabla A_j^{(0)})_3]\Bigg]. \end{aligned} \tag{24}$$

Now, we analyze the different terms in (24); the first one in the exponential is the usual holonomy (corrected at order θ) which classifies the differents homotopy classes.

The term

$$\int_{x_0}^{x} dx_j \; [\mathbf{A^{(0)}} \times \nabla A_j^{(0)}]_3, \tag{25}$$

is a non-commutative correction to the vortex decaying as $1/r^3$ and does not contribute to the line spectrum.

Finally, the term

$$\theta\int_{x_0}^{x} dx_j \,[(\mathbf{v}\times\nabla A_j^{(0)}]_3 \tag{26}$$

is a velocity dependent correction insensitive to the topology of the manifold. If $\mathbf{v}$ increases, the contribution of (26) to the holonomy oscillates very quickly smearing the fringe pattern. The above result is completly different to the commutative Aharonov-Bohm effect where the fringe pattern is insensitive to the relativistic nature of the electrons [6].

III. POSSIBLE EXPERIMENTAL STRATEGY FOR DETECTING NONCOMMUTATIVE EFFECTS

In this section we will analyse the possibility of detecting spatial noncommutative effects by using the Tonomura experiment. Our claim is that if one increase the energy of the incident electrons beam, then one should observe a squeezing of the fringes pattern. As we expect that θ is small, then θ –as was assumed in the previous section– is a perturbative parameter and the main part of the fringe pattern is contained in the commutative sector of the Aharonov-Bohm.

If we now modify the energy of the electrons beam, then we should see noncommutative corrections. In the original Tonomura experiment [10], they consider incident beams with energies among 80, 100 and 125 keV but they did not find changes in the fringes pattern. In our opinion, this is explained by the perturbative stability of the Aharonov-Bohm fringes pattern but, if we can consider incident energies among 160-500 keV (or higher) spectral differences should be observed. At this energies the effects of the penetrability can be important, but this fact does not affect the flux-dependent phase difference [11].

Presently, we have several bounds for θ [12]. The noncommutative contributions will, in general, be tiny. We hope, however, that futureTonomura or improved Tonomura kind experiments will be able to distinguish these new effects.

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IV. APPENDIX

In this appendix, we would like to discuss some implications of the relativistic Aharonov-Bohm effect. In particular, we would like to emphasize that due the topological character of the Aharanov-Bohm effect (and also to the fact that the radius of the solenoid is zero) the interference pattern does not change by relativistic corrections.

This last fact can be seen as follows; using the reference [6] one see that the Green function associated to the usual Aharonov-Bohm effect is given by

$$G[x,x^{'}] = \sum_{n=-\infty}^{\infty} (-i)^{|n+\phi|}\exp[-i(n+\phi)]F_{|n+\phi|}, \tag{27}$$

where ϕ is the magnetic flux and the function $F_{|n+\phi|}$ for the non-relativistic case is

$$F_{|n+\phi|} = \frac{m}{2\pi i}\exp[\frac{2mi}{\tau}(r^2+r^{'2})]J_{|n+\phi|}(\frac{mrr^{'}}{\tau}), \tag{28}$$

where $\tau = t - t^{'}$ and J_α are Bessel functions. For the relativistic case the calculation is similar. Indeed, after using the proper-time gauge the function $F_{|n+\phi|}$ becomes

$$\begin{aligned} F_{|n+\phi|} = \\ \int d^2p\int_0^\infty d\lambda \exp[\; i\;\; p_\mu \Delta x^\mu - \frac{\lambda}{2}(p^2+m^2)]J_{|n+\phi|}(\frac{rr^{'}}{\lambda}). \end{aligned} \tag{29}$$

where $\lambda = N(0)(t-t^{'})$ with $N(0)$ the einbein.

If we use the Poisson summation formula, then in both the relativistic as well as in the non relativistic case, the Green function is

$$G[x,x^{'}] = \sum_{n=-\infty}^{\infty} e^{2i\pi n\phi} K_n, \tag{30}$$

where K_n is defined as

$$K_n = \int_{-\infty}^{\infty} d\omega \, (-i)^{|\omega|} \, e^{-i\omega\phi} \, F_{|\omega|}, \tag{31}$$

and, as a consequence, the wave function becomes

$$\psi(x) = \sum_{n=-\infty}^{\infty} e^{2i\pi n\phi} \varphi_n(x), \tag{32}$$

with

$$\varphi_n(x) = \int dy \; G_n \,[x,y]\, \psi(y), \tag{33}$$

being φ_n and $G_n[x,y]$, respectively, the wave and Green functions for the n-th homotopy class [9].

Thus, from (32) one see that the relativistic character of the system is contained in K_n and only the exponential factor, which does not depend on the energy, is responsible for the fringe pattern. This result reflects the topological nature of the commutative Aharonov-Bohm effect. However, our formula (26) show us that the non-commutative Aharonov-Bohm effect is radically different because the fringe pattern must change when the electrons are getting higher energies.

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